CHOICE OF A MATHEMATICAL MODEL FOR AN UNSTEADY HEAT TRANSFER PROCESS WITH A SINGLE-PHASE INCOMPRESSIBLE HEAT TRANSFER AGENT

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An examination is made of the possible error introduced by neglecting the nature of the radial temperature distribution when calculating the temperature field in a cylindrical heating shell.

In mathematical modeling of unsteady heat transfer in a single-phase incompressible section, it is important to know the possible errors introduced by a simplified examination of the process of accumulation of heat in the metal of the heating element. In the majority of cases, when examining the dynamic characteristics of this kind of heat-exchange equipment, we neglect the effect of the radial temperature distribution in the metal of the heating element [1–3]. For fixed relationships of the technical thermal parameters and the geometric dimensions of the heater, this examination may give an appreciable dynamic error, especially in the high-frequency region. On the other hand, high accuracy of calculation is not required in a number of cases, and complication of the mathematical model is not necessary.

In this paper an examination is made, on the example of a hollow cylindrical heating shell, of the expediency of using the exact and the approximate solutions, and an estimate is made of the possible error.

We shall examine the derivation of the exact transfer function for the section under consideration (neglecting heat conduction of the metal and of the heat transfer agent in the axial direction, and heat conduction of the agent in the radial direction), which will allow us to make a valid determination of the conditions when it is possible to simplify the mathematical model by replacing the heat conduction equation by a heat balance equation for the metal of the heating element.

It is known that the dynamic characteristics of a single-phase section, a heating shell of finite length l with a finite value of the ratio of outer to inner diameter, $R = d_e/d_i$, washed inside by an incompressible heat transfer agent, are determined by two characteristic times: the time for a particle of the agent to traverse the circuit of the section, τ_2, and a time constant, τ_1, which describes the intensity of heat transfer from the surface of the element to the agent.

For a single-phase incompressible fluid

$$\tau_2 = \frac{l\gamma d_i^2}{G}\frac{\pi}{4} \qquad (1)$$

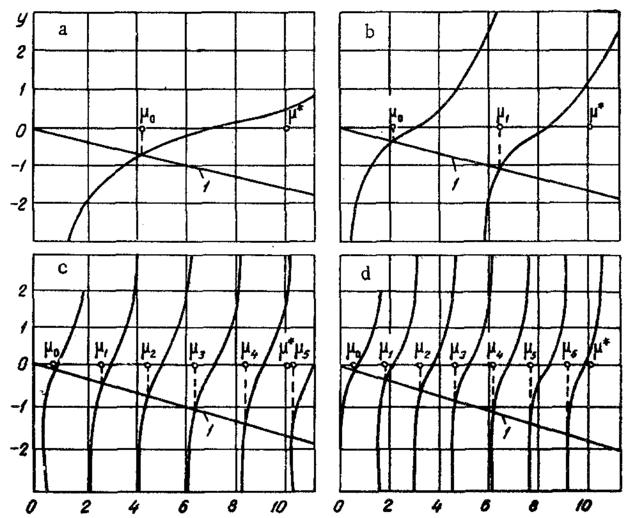
is determined in an elementary way from the relations describing the equilibrium state of the process and appearing in the coefficients of the heat balance equation for the heat transfer agent

$$C\gamma d_i^2 \frac{\pi}{4}\frac{\partial t}{\partial \tau} + GC\frac{\partial t}{\partial x} = \alpha p(t_{\text{м}}|_{d_i} - t) \qquad (2)$$

with the boundary condition at x = 0

$$t = t_{in}.$$

Equation (2) was obtained rigorously, if it is considered that the velocity of the agent is the same over the cross section of the channel. In other words, this is a definite approximation, just like the introduction of the heat transfer coefficient in the unsteady regime.



Graphical interpretation of the equation
$$\frac{I_0(\mu)N_1(\mu R) - I_1(\mu R)N_0(\mu)}{I_1(\mu)N_1(\mu R) - N_1(\mu)I_1(\mu R)} = -\frac{\mu}{\mathrm{Bi}}$$
for determining the eigenvalues μ_n for a hollow cylindrical geometry with various values of the geometric parameter R: a) R = 1.2; b) 1.5; c) 2.5; d) 3.0; 1) $y_1 = -\mu/\mathrm{Bi}$.

In turn, τ_1 is characterized by the coefficient of heat transfer between the surface of the heating element and the medium being heated, and also by the heat conduction properties of the metal of the heating element, and is formally defined as the time constant of the exponential approximation to the timewise variation of the temperature of the internal surface of the heating element, when there is a discontinuous change in the power of the internal heat sources.

For a constant value of the thermal conductivity λ we have the equation of heat conduction with internal heat sources, written in dimensionless form,

$$\frac{\partial \nu}{\partial \tau} = \mathrm{Fo}\,\Delta\nu + q - \frac{\partial \Theta}{\partial \tau} \qquad (3)$$

with the boundary and initial conditions

$$\frac{1}{\mathrm{Bi}}\frac{\partial \nu}{\partial N} + \nu|_{d_i} = 0,$$

$$\left.\frac{\partial \nu}{\partial N}\right|_{d_e} = 0, \quad \nu|_{\tau=0} = 0, \tag{4}$$

where

$$\nu = T - \Theta = \frac{t_м - t_{м_0}}{q_{V_0}(\tau_2/C_м \gamma_м)} - \frac{t - t_0}{q_{V_0}(\tau_2/C_м \gamma_м)},$$

$$q = \frac{q_V - q_{V_0}}{q_{V_0}}; \qquad \tau = \frac{\tau}{\tau_2}$$

and d_i is the characteristic geometric dimension for the section of the heater shell.

We shall seek a solution of Eq. (3) in the form:

$$\nu = \sum_{n=0}^{\infty} \nu_n(\tau) X_n(r), \tag{5}$$

where $X_n(r)$ is an eigenfunction of Eq. (3) with boundary conditions (4) corresponding to the n-th eigenvalue of the parameter μ_n, and is determined from the equations

$$\Delta X(r) + \mu^2 X(r) = 0, \tag{3*}$$

$$\frac{1}{\mathrm{Bi}}\frac{\partial X(r)}{\partial N} + X(r)|_{d_i} = 0,$$

$$\left.\frac{\partial X(r)}{\partial N}\right|_{d_e} = 0, \tag{4*}$$

and $r = r/d_i$ is a dimensionless coordinate.

The solutions of Eq. (5) corresponding to different eigenvalues are mutually orthogonal in the range $1 \leq r \leq R$ with weight $\rho(r) = r$:

$$\int \rho(r) X_m(r) X_n(r)\, dr = 0 \text{ when } m \neq n,$$

and the integration is performed over the region in which the solution is sought.

We shall write the expression $q - \frac{\partial \Theta}{\partial \tau}$ in the form of an expansion in terms of the functions $X_n(r)$:

$$q - \frac{\partial \Theta}{\partial \tau} = \left(q - \frac{\partial \Theta}{\partial \tau}\right)\sum_{n=0}^{\infty} B_n X_n(r), \tag{6}$$

where $\sum_{n=0}^{\infty} B_n X_n(r) = 1$ is the expansion of a function identically equal to 1 in the functions $X_n(r)$, while

$$B_n = \frac{\int \rho(r) X_n(r)\, dr}{\int \rho(r) X_n^2(r)\, dr}.$$

Substituting Eqs. (6) and (5) into Eq. (3), we obtain

$$\sum_{n=0}^{\infty}\left\{\frac{\partial \nu_n}{\partial \tau} + \mathrm{Fo}\,\mu_n^2 \nu_n - \left(q - \frac{\partial \Theta}{\partial \tau}\right) B_n\right\} X_n(r) = 0.$$

This equality, like the initial condition $\nu|_{\tau=0} = 0$, must be satisfied identically for any r. For this it is necessary that each function ν_n satisfy the equation

$$\frac{\partial \nu_n}{\partial \tau} + a_n \nu_n = B_n\left(q - \frac{\partial \Theta}{\partial \tau}\right), \quad \nu_n|_{\tau=0} = 0, \tag{7}$$

where $a_n = \mathrm{Fo}\,\mu_n^2$.

Solving Eq. (7), we find

$$\nu_n = B_n \int_0^{\tau}\left\{\left(q - \frac{\partial \Theta}{\partial \tau}\right)\exp\left[-a_n(\tau - \tau')\right]\right\} d\tau'. \tag{8}$$

Thus, if ν_n has the form (8), then Eq. (5), when it satisfies Eq. (3) and the initial condition, will also satisfy the boundary conditions, since all the $X_n(r)$ satisfy them.

Therefore the solution of Eq. (3) is

$$\nu = \sum_{n=0}^{\infty} B_n X_n(r) \int_0^{\tau}\left\{\left(q - \frac{\partial \Theta}{\partial \tau}\right)\exp\left[-a_n(\tau - \tau')\right]\right\} d\tau'. \tag{9}$$

Examining Eq. (9) on the inner surface of the shell, we may write

$$\nu|_{d_i} = \sum_{n=0}^{\infty} A_n \int_0^{\tau}\left\{\left(q - \frac{\partial \Theta}{\partial \tau}\right)\times \exp\left[-a_n(\tau - \tau')\right]\right\} d\tau', \tag{10}$$

where

$$A_n = B_n X_n(r)|_{d_i}.$$

Expression (10) may be regarded as an equation that relates the temperature $T = \nu + \Theta$ of the shell to that of the heat transfer agent Θ.

We write Eq. (2) in the dimensionless form

$$\frac{1}{\omega}\frac{\partial \Theta}{\partial \tau} + \frac{\partial \Theta}{\partial x} = k\nu|_{d_i} \tag{11}$$

with the initial and boundary conditions

$$\Theta|_{\tau=0} = 0,$$

$$\Theta|_{x=0} = \Theta_{in}, \tag{12}$$

where $k = \alpha\, pl/GC$. Eliminating the variable $\nu|_{d_i}$ from Eqs. (10) and (11), we obtain the following equation relating the unknown Θ with the initial and boundary conditions (12):

$$\frac{\partial \Theta}{\partial x} + \frac{1}{\omega}\frac{\partial \Theta}{\partial \tau} = \sum_{n=0}^{\infty} kA_n \int_0^{\tau}\left\{\left(q - \frac{\partial \Theta}{\partial \tau}\right)\times \exp[-a_n(\tau - \tau')]\right\} d\tau'. \tag{13}$$

Following a Laplace transformation of Eq. (13) with respect to the variable τ and integration with respect to the space coordinate x from 0 to 1, we obtain the transfer functions for the temperature of the agent at the outlet from the heated section for perturbations of the power of the internal heat sources and the inside emperature:

$$W(S)_{\bar{\Theta}_{out},\, \bar{q}} = \frac{k\sum_{n=0}^{\infty}(A_n/[S + a_n])}{S\left[1 + k\sum_{n=0}^{\infty}(A_n/[S + a_n])\right]} \times$$

$$\times\left\{1-\exp\left[-S\left(1+k\sum_{n=0}^{\infty}\frac{A_n}{S+a_n}\right)\right]\right\}, \quad (14)$$

$$W(S)_{\bar{\theta}_{out},\,\bar{\theta}_{in}}=\exp\left[-S\left(1+k\sum_{n=0}^{\infty}\frac{A_n}{S+a_n}\right)\right]. \quad (15)$$

The terms

$$\exp[-S],$$

$$\exp\left[-k\sum_{n=0}^{\infty}\frac{A_n S}{S+a_n}\right] \quad (16)$$

in (15) for the dimensionless time to the scale τ_2 determine in the Laplace transformation the time for a particle of the agent to traverse the section, and the distortion of the shape of the perturbing input signal by the accumulation of heat in the walls of the heating element, respectively.

If we write the expression for the series appearing in the exponent of Eq. (16) in the form

$$k\sum_{n=0}^{\infty}\left[\frac{A_n}{a_n}S\Big/\left(\frac{1}{a_n}S+1\right)\right], \quad (17)$$

since the relations

$$\frac{A_n}{a_n}\sim\frac{1}{\mu_n^4} \text{ and } \frac{1}{a_n}\sim\frac{1}{\mu_n^2}$$

hold, it is not difficult to show that for large values of n the coefficients of S examined above will tend to quantities of a higher order of smallness than μ_n.

It may be seen from the figure (p. 305) that with increase of R (corresponding to increase in the wall thickness of the heating element), the period of the function y, which determines the values of the characteristic roots of the system (3*), (4*) for the value of the Bi number under consideration, decreases. In other words, with increase of R, the value of the first and of all the successive roots $\mu_0, \mu_1, \mu_2, \ldots, \mu_n$ for R_i will be less than the values of the corresponding roots for R_{i-1}, where $R_i > R_{i-1}$.

If we consider that for the required accuracy of the dynamic calculations it is sufficient to restrict ourselves to the value of the root μ^*, it is then evident that the number of terms of the series (17) will increase with increase of R.

Therefore, for values of R close to unity, we may restrict ourselves, with sufficient accuracy, to the first term of series (17), which corresponds physically to neglecting the distribution of the temperature of the metal of the heating shell over the radius, or, expressed analytically, the heat conduction equation is replaced by a heat balance equation of the form

$$q(x)-\alpha p(t_m-t)=C_m\gamma_m\frac{\pi(d_e^2-d_i^2)}{4}\frac{\partial t_m}{\partial\tau}. \quad (18)$$

In this simplified representation, the transcendental component of the transfer functions (14) and (15) takes the form:

$$\exp\left\{-\left[\tau_2 S+\tau_1 S\Big/\left(\frac{\tau_1}{k}S+1\right)\right]\right\}, \quad (19)$$

where

$$\tau_1=\frac{C_m(d_e^2-d_i^2)\gamma_m}{\alpha p}\frac{\pi}{4}, \quad k=\frac{\alpha pl}{GC}.$$

It is a straightforward matter to verify that the solution of this simplified problem, i.e., the simultaneous solution of Eqs. (2) and (18), is a special case of the general problem when $R\to 1$.

In fact, by representing the expression for the first term of the series (17) in the real time scale

$$\frac{k\tau_2 A_0}{a_0}S\Big/\left(\frac{\tau_2}{a_0}S+1\right), \quad (20)$$

it may be shown that the coefficients of S in Eq. (19) and (20) are identically equal when $A_0 = 1$, which corresponds to R = 1 with the condition

$$\sum_{n=0}^{\infty}B_n X_n(r)=1.$$

In a number of cases, when examining this type of unsteady problem, the heat balance Eq. (18) may be used, allowing for the thermal resistance of the shell metal in the heat transfer coefficient

$$\alpha^*=1\Big/\left(\frac{1}{\alpha}+\frac{d_e}{2\lambda}\ln R\right).$$

Then the degree of approximation to the exact solution (it may be determined from the values of Eqs. (19) and (16) when $S\to\infty$) will be given by the expression

$$1-\exp(k_1-k),$$

where

$$k_1=\alpha^* pl/GC;$$

on the other hand, the value

$$\lim_{S\to\infty}\exp\left[-k\sum_{n=0}^{\infty}\frac{A_n S}{S+a_n}\right]=\exp[-k]$$

allows us to determine the number of terms of the series which will provide the given accuracy of solution in this case.

In approximate investigations for heating elements with large values of R, it is expedient to represent the transcendental component in the form

$$\prod_{n=0}^{\infty}\frac{1}{(kA_n/a_n)S+1}, \quad (21)$$

where

$$\frac{1}{(kA_i/a_i)S+1} \quad (0<i<\infty)$$

is the approximation expression

$$\exp\left[-\frac{kA_iS}{S+a_i}\right].$$

In practical calculations it is necessary to limit the series, in which case the accuracy obtained is of the order of the value of the coefficient of the last term of Eq. (21).

The theoretical premises associated with the choice of mathematical models of heat-exchange equipment have been subjected to experimental verification and were applied in [4], which gives a theoretical and experimental determination of the dynamic characteristics of a technical model of the Kurchatov Beloyarsk atomic power station.

The points examined in this article permit the choice of a rational mathematical model of heat-exchange equipment with predetermined accuracy.

The technique described also extends to mathematical models of unheated parts such as connecting tubes, separator and evaporator drums, and water-moderated water-cooled reactor vessels.

NOTATION

γ, γ_m are the specific weight of heat transfer agent and metal, respectively; ω dimensionless heat transfer agent velocity; t, t_m are the temperature of heat transfer agent and metal, respectively; α is the heat transfer coefficient; G is the mass flow rate of heat transfer agent; p is the perimeter washed by heat transfer agent; C, C_m are the specific heat of heat transfer agent and metal, respectively; q_v, q(x) are the specific energy output referred to unit volume and unit length, respectively; $\partial/\partial N$ is the derivative with respect to direction of outward normal to surface; S is the Laplace transformation constant.

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